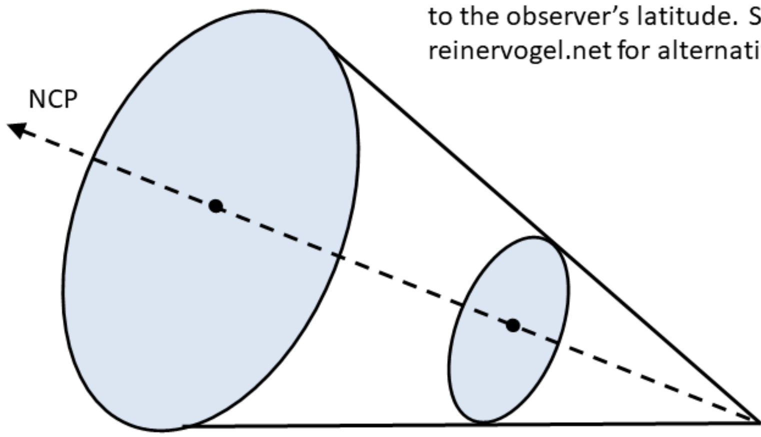
Mathematical Models of the VNS Equatorial Platform

Background

Equatorial platforms provide a way for Dobsonian telescopes to track the sky for a limited amount of time. A common design is the Gee equatorial platform that uses sectors that follow a conic envelope. When the cone axis points at the NCP the Dobsonian will track the sky for as long as the size of the sectors allows it.

"Gee" type: Build a platform with one or two circular bearings (sectors) that follow a conic envelope with the rotation axis pointing at the NCP. The sectors are tilted at an angle equal to the observer's latitude. See reinervogel.net for alternative designs.



The side view of a Gee design shows the two tilted cone sections that intersect with the plane in which the bearings that carry the sectors are placed. The sectors carry the platform base (not shown) and stick out underneath that plane. To minimize the torque required to move the platform, the center of mass of the top assembly including the Dobsonian, is placed on the cone axis.

The advantages of the Gee design are (1) its conceptual simplicity, (2) tracking the sky exactly with a constant motor speed that drives the sectors, and (3) the flexibility in vertical placement of the cone axis to accommodate the center of mass. The disadvantage are (1) the tilting of the sectors that require more work, (2) bearings on both sides of the sectors and (3) it is harder to design the motor drive. The Gee design works best at lower latitudes. At higher latitudes the bearings would be tilted too much and other designs such as Poncet or d'Autumework better.

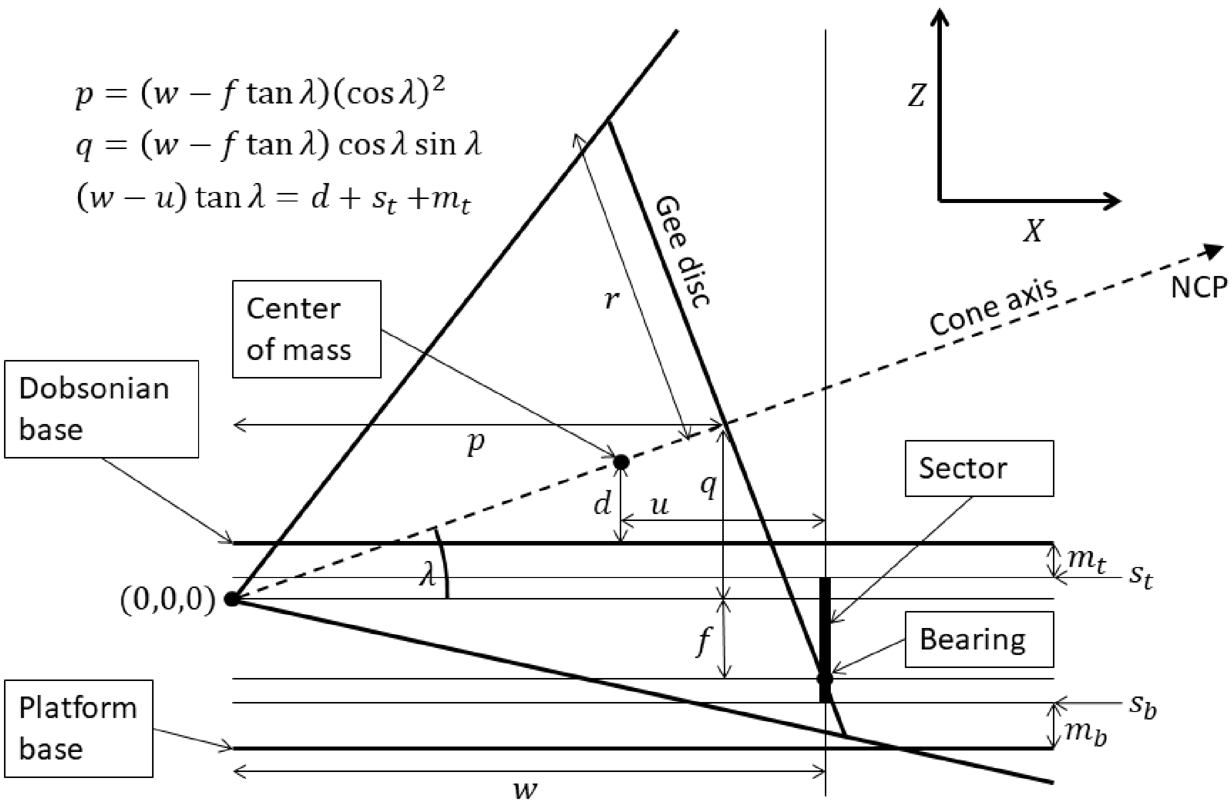
The VNS (Vertical North Segments) Design

The VNS design uses two vertical sector sectors at the North side. The advantages are that (1) the sectors are easier to make and (2) the vertical segments are mechanically more stable than Gee sectors. The disadvantages are (1) more mathematical complexity, (2) the sectors must either be very thin or conically shaped edges and (3) the motor speed is not constant.

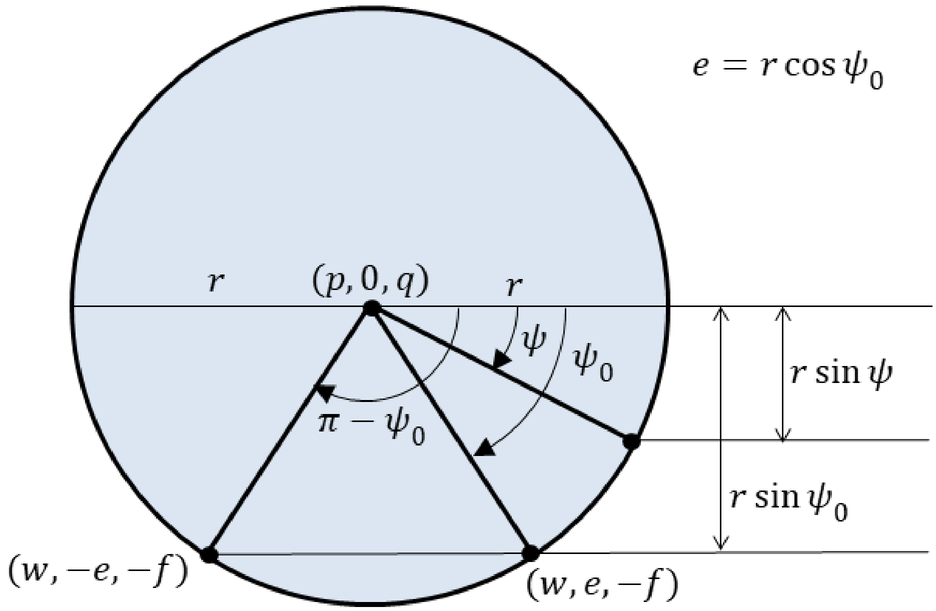
In this section we will solve the mathematical design of the sectors. The parameters are related to the cone and bearing locations. Later on we will define a more user-friendly set of design parameters.

Below is a side view of the VNS design. Let us summarize the features that are displayed.

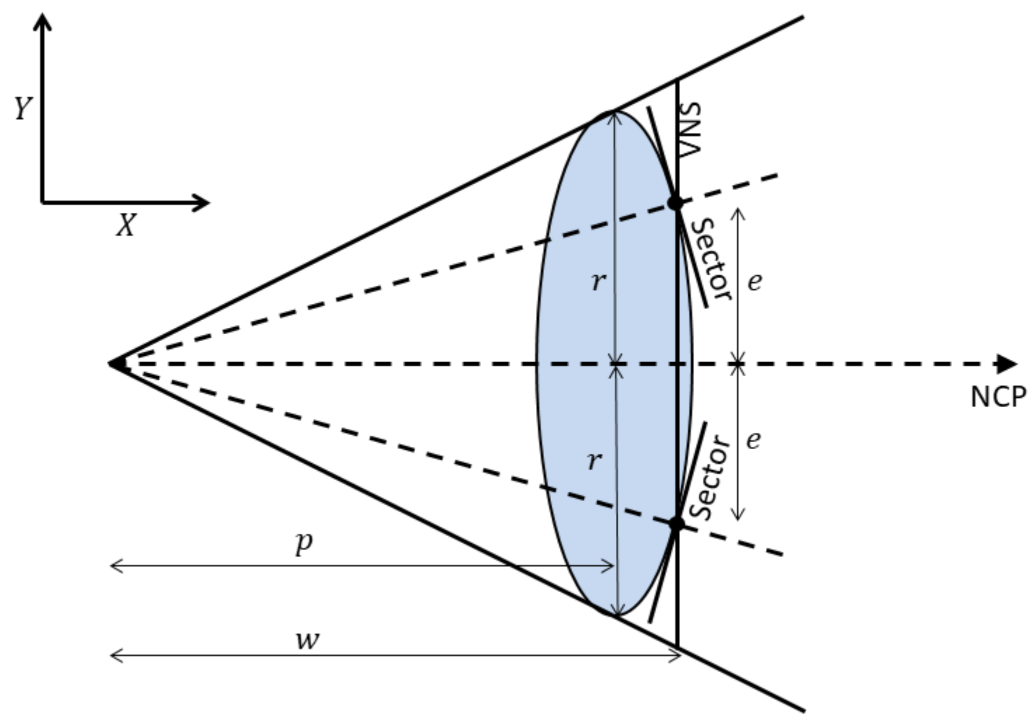
The



The cone is shown in fat lines with an axis that points at the NCP with an angle equal to the latitude of the observer. Its apex is the origin of our right-handed XYZ coordinate system. The horizontal fat lines are the (bottom of the) Dob base and the (top of the) bottom board of the platform. In between it are the sectors that ride on the bearings. The top sector margin m must leave room for mechanical implementation, and the bottom sector margin b must be large enough for the sector not to hit the platform base as it rotates. The center of mass is on the cone axis and is a distance d above the Dob base. It is set back by a distance u relative to the North bearings. The Gee disc has radius r and its center is placed at a horizontal distance p and a vertical distance q from the apex. Below is a figure of the Gee disc as seen from the North side along the cone axis. The bearings are a horizontal distance e away from the cone apex.



The top view in the figure below shows the tilted Gee disc and the VNS sectors. The sectors are placed in the tangent of the projection of the disc on the XY plane.



Building a VNS Platform with a Jig

The most practical method is to use a jig where a belt sander or router shapes the VNS sectors. The jig is a device that rotates the platform top around its virtual axis and shapes the sectors at the point of contact with the bearings. Starting with a rough sector profile that is a bit larger than the final shape, the belt sander is moved in while rotating the platform until it is at the bearing location.

Building a VNS platform with a jig requires no math other than calculating the location of the virtual axis. Clearly, making a good jig is not easy, and it may not be practical for everyone. We will therefore do the equivalent math instead so we can print the sector shape and cut it with a jigsaw from aluminum or wood. The mathematical design assumes zero thickness of the sector, so aluminum is a better choice than triplex or other kinds of wood.

The mathematical design consists of the following "back-and-forth" algorithm:

- Place the top board in the nominal position (horizontal / level)
  - The sector is assumed to be in a vertical plane in this position
  - The bearing is at an angle ψ_0
- Choose a point on the sector profile specified by an angle of rotation $\psi - \psi_0$
  - The coordinates of the profile point are not yet known
- Rotate the sector to the bearing axis over an angle $-(\psi - \psi_0)$
- Calculate the intersection of the rotated sector plane with the bearing axis
  - This yields the coordinates of the profile point in the rotated position
- Rotate the sector back to the nominal position over an angle $\psi - \psi_0$
  - The profile point coordinates are calculated by applying the rotation math

Doing this for all points will generate the points of the profile. Note that there is a shortcut if the bearing axis coincides with the cone surface, namely, when $f = 0$. In that case there is no point in rotating back and forth, and we can simply calculate the profile as the intersection of the sector plane with the cone. In all other cases when f is nonzero we must do the back-and-forth algorithm.

Sector Calculations

The figure contains the formulas that show how to derive w, p, q from f, λ, d, m_t and s_t. These parameters are user defined except for s_t that depends on the run time T. As the platform rotates from start to reset, the sector angle ψ will go from $\psi_0 - \psi_T$ through $\psi_0 + \psi_T$, where ψ_T is derived directly from T (4 minutes = 1 degree). We repeat these formulas here:

$$p = (w - f \tan \lambda)(\cos \lambda)^2$$

$$q = (w - f \tan \lambda) \cos \lambda \sin \lambda$$

$$(w - u) \tan \lambda = d + m_t + s_t$$

With the origin at the apex, let the Gee disc be described by

$$s(\psi) = \begin{pmatrix} p \\ 0 \\ q \end{pmatrix} + r \begin{pmatrix} \cos\lambda & 0 & -\sin\lambda \\ 0 & 1 & 0 \\ \sin\lambda & 0 & \cos\lambda \end{pmatrix} \begin{pmatrix} 0 \\ \cos\psi \\ -\sin\psi \end{pmatrix} = \begin{pmatrix} p \\ 0 \\ q \end{pmatrix} + r \begin{pmatrix} \sin\lambda\sin\psi \\ \cos\psi \\ -\cos\lambda\sin\psi \end{pmatrix}$$

where $0 \le \psi \le 2\pi$ is the local coordinate of a unit vector in the YZ plane where $\psi = 0$ points West. We have chosen the axes such that our XYZ coordinate system is right handed, the positive X axis points North and ψ is such that a platform in the Northern hemisphere will track the sky for increasing ψ, namely, a clockwise rotation over the angle ψ as seen from the NCP. To find the bearing location angle $\psi = \psi_0$ we set $z = -f$:

$$q - r\cos\lambda\sin\psi_0 = -f$$

This will give us

$$\psi_0 = \sin^{-1}\frac{q+f}{r\cos\lambda}$$

And the bearing location

$$s(\psi_0) = \begin{pmatrix} w \\ e \\ -f \end{pmatrix} = \begin{pmatrix} w \\ r\cos\psi_0 \\ -f \end{pmatrix}$$

We can use the Gee sector formula for the vertical sector coordinates because those of the VNS sector will be close enough. The sector top and bottom are then given by

$$s_t = q - r\cos\lambda\sin(\psi_0 - \psi_T)$$

$$s_b = q - r\cos\lambda\sin(\psi_0 + \psi_T)$$

In the nominal position, the sector plane is vertical and tangent to the Gee disc at the bearing location when viewed from above. The tangent at the bearing is given by

$$\frac{\partial s}{\partial\psi}(\psi_0) = r \begin{pmatrix} \cos\lambda & 0 & -\sin\lambda \\ 0 & 1 & 0 \\ \sin\lambda & 0 & \cos\lambda \end{pmatrix} \begin{pmatrix} 0 \\ -\sin\psi_0 \\ -\cos\psi_0 \end{pmatrix} = r \begin{pmatrix} \sin\lambda\cos\psi_0 \\ -\sin\psi_0 \\ -\cos\lambda\cos\psi_0 \end{pmatrix}$$

Projecting on the XY plane and normalizing, the normalized tangent is the vector

$$\begin{pmatrix} x_s \\ y_s \\ 0 \end{pmatrix} = \frac{r\begin{pmatrix} \sin\lambda\cos\psi_0 \\ -\sin\psi_0 \\ 0 \end{pmatrix}}{\left\| r\begin{pmatrix} \sin\lambda\cos\psi_0 \\ -\sin\psi_0 \\ 0 \end{pmatrix} \right\|}$$

To calculate the sector profile, let us first assume that $f = 0$ in which case all we have to do is to calculate the intersection of the cone with the sector plane. The line from the apex through a point $s(\psi)$ on the circle is of the form $s(\psi)\sigma(\psi)$. If we define

$$X = \begin{pmatrix} x_s & 0 \\ y_s & 0 \\ 0 & 1 \end{pmatrix}$$

then the intersection of that line with the sector plane is determined by

$$s(\psi)\sigma(\psi) = \begin{pmatrix} w \\ e \\ 0 \end{pmatrix} + X \begin{pmatrix} \mu(\psi) \\ \nu(\psi) \end{pmatrix}$$

Here, $\mu(\psi)$ and $\nu(\psi)$ are the profile coordinates in the sector plane that describe the shape of the sector. This can be solved by a simple inversion from

$$\begin{pmatrix} s(\psi) & -X \end{pmatrix} \begin{pmatrix} \sigma(\psi) \\ \mu(\psi) \\ \nu(\psi) \end{pmatrix} = \begin{pmatrix} w \\ e \\ 0 \end{pmatrix}$$

The coordinates $\mu(\psi)$ and $\nu(\psi)$ describe the sector profile in a local 2D coordinate system.

For the case when f is nonzero we must perform the back-and-forth algorithm, also referred to as the milling algorithm. Let us define the clockwise rotation over an angle φ as seen from the NCP around the cone axis as $B(\varphi)$. This is a rotation of the cone axis down to the X axis followed by a rotation around the X axis followed by a rotation of the X axis back to the cone axis:

$$B(\varphi) = \begin{pmatrix} \cos\lambda & 0 & -\sin\lambda \\ 0 & 1 & 0 \\ \sin\lambda & 0 & \cos\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & \sin(\varphi) \\ 0 & -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{pmatrix}$$

Note that $B(\varphi)$ is an orthogonal matrix, so $B(-\varphi) = B'(\varphi)$. Let us assume that the sector is in the nominal position, meaning vertically aligned with the Dobsonian base horizontal and the segment centered on the bearing. The profile in this position is of the form

$$\xi(\psi) = \begin{pmatrix} w \\ e \\ 0 \end{pmatrix} + X \begin{pmatrix} \mu(\psi) \\ \nu(\psi) \end{pmatrix}$$

where $\mu(\psi)$ and $\nu(\psi)$ are the horizontal and vertical coordinates in the local 2D coordinate system of the sector. To find the value of these coordinates for a profile point $\xi(\psi)$ we first rotate the sector over an angle $\psi_0 - \psi$ to place the profile point on the bearing axis:

$$B(\psi_0 - \psi)\left(\begin{pmatrix} w \\ e \\ 0 \end{pmatrix} + X \begin{pmatrix} \mu(\psi) \\ \nu(\psi) \end{pmatrix}\right) = \begin{pmatrix} w \\ e \\ 0 \end{pmatrix}\sigma(\psi) + \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix}$$

This is solved from the linear equation

$$\begin{pmatrix} \begin{pmatrix} w \\ e \\ 0 \end{pmatrix} & -B(\psi_0 - \psi)X \end{pmatrix} \begin{pmatrix} \sigma(\psi) \\ \mu(\psi) \\ \nu(\psi) \end{pmatrix} = B(\psi_0 - \psi)\begin{pmatrix} w \\ e \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}$$

Pre-multiplying this with $B(\psi)$ avoids the transpose, which is perhaps more convenient,

$$\begin{pmatrix} B(\psi-\psi_0)\begin{pmatrix} w \\ e \\ 0 \end{pmatrix} & -X \end{pmatrix}\begin{pmatrix} \sigma(\psi) \\ \mu(\psi) \\ \nu(\psi) \end{pmatrix} = \begin{pmatrix} w \\ e \\ 0 \end{pmatrix} + B(\psi-\psi_0)\begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}$$

With $\sigma(\psi)$, $\mu(\psi)$ and $\nu(\psi)$ known, we then rotate the sector forward back to where the profile point was taken,

$$\xi(\psi) = B(\psi-\psi_0)\left(\begin{pmatrix} w \\ e \\ 0 \end{pmatrix}\sigma(\psi) + \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix}\right) = \begin{pmatrix} w \\ e \\ 0 \end{pmatrix} + X\begin{pmatrix} \mu(\psi) \\ \nu(\psi) \end{pmatrix}$$

By doing this calculation for each profile point the entire sector profile is generated both in 2D and 3D.

Angular Velocity

As was mentioned above, when the sectors move away from the nominal position, the point of contact with the bearings shifts slightly outward. Also, the angle between axis and sector tangent will shift away slightly from 90 degrees. Both effects have an impact on the angular velocity of the platform. Ideally, the motor control must compensate such that these effects are eliminated.

Let us define ω_0 as Earth's angular velocity. We design the platform in first instance such that the motor rotates at a speed ω_M such that the platform counter-rotates at the same angular velocity. This is accomplished by friction at the contact between bearing and sector.

First, let us look at the varying distance from the cone apex to the bearing contact as the platform rotates. It should be clear that the angular velocity of the platform is inversely proportional to this distance. If we normalize it to be 1 in the nominal position then this ratio is $\sigma(\psi)$. The motor control should compensate for this by setting the motor speed inversely proportional to this function.

Secondly, let us look at the angle between the sector tangent and the tangent to the axis at the bearing contact. There are two forces that act on the sector: (1) a tangential force from the motor for tracking the sky, and (2) a radial force due to the geometry that moves the sector contact outward and inward as the platform rotates. The latter is parallel to the axis, which is nearly perpendicular to the tangent surface and the tangential force. Because of that, we may ignore it for the effect on the tangential rotation speed. There are bizarre surface properties that could make it relevant, such as grooves in the direction of the bearing axis versus grooves perpendicular to the sector that could make a difference. A normal steel-to-aluminum interface does not have grooves, so we can ignore such cases. However, an ill-balanced system could cause slippage at a nonzero angle with the bearing axis, which equates to grooves in that direction, and it could become relevant in that case.

Despite the fact that the angle is largely irrelevant, we have done the math for finding the derivative and its radial component, so we can analyze the two bizarre interface cases just for the record, as irrelevant as it may be.

We will need the derivative of the rotation matrix, which is

$$\dot{B}(\varphi) = \begin{pmatrix} \cos\lambda & 0 & -\sin\lambda \\ 0 & 1 & 0 \\ \sin\lambda & 0 & \cos\lambda \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin(\varphi) & \cos(\varphi) \\ 0 & -\cos(\varphi) & -\sin(\varphi) \end{pmatrix} \begin{pmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\sin\lambda \\ 1 & 0 \\ 0 & \cos\lambda \end{pmatrix} \begin{pmatrix} -\sin(\varphi) & \cos(\varphi) \\ -\cos(\varphi) & -\sin(\varphi) \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{pmatrix}$$

Here, the dot means the derivative w.r.t. φ. Let us take the derivative of the solution equation,

$$\dot{B}(\psi-\psi_0)\begin{pmatrix} w \\ e \\ 0 \end{pmatrix}\sigma(\psi) + \left(B(\psi-\psi_0)\begin{pmatrix} w \\ e \\ 0 \end{pmatrix} \quad -X \right) \begin{pmatrix} \dot{\sigma}(\psi) \\ \dot{\mu}(\psi) \\ \dot{\nu}(\psi) \end{pmatrix} = \dot{B}(\psi-\psi_0)\begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}$$

We can solve the derivatives of the local parameters from

$$\left(B(\psi-\psi_0)\begin{pmatrix} w \\ e \\ 0 \end{pmatrix} \quad -X \right) \begin{pmatrix} \dot{\sigma}(\psi) \\ \dot{\mu}(\psi) \\ \dot{\nu}(\psi) \end{pmatrix} = \dot{B}(\psi-\psi_0)\left(\begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} - \begin{pmatrix} w \\ e \\ 0 \end{pmatrix}\sigma(\psi) \right)$$

To find $\dot{\xi}(\psi)$ we first have to take the derivative of the second expression,

$$\dot{\xi}(\psi) = B(\psi-\psi_0)\left(\begin{pmatrix} w \\ e \\ 0 \end{pmatrix}\sigma(\psi) + \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix} \right) = \dot{B}(\psi-\psi_0)\begin{pmatrix} w \\ e \\ 0 \end{pmatrix}\sigma(\psi) + B(\psi-\psi_0)\begin{pmatrix} w \\ e \\ 0 \end{pmatrix}\dot{\sigma}(\psi)$$

Note that this is the derivative with the sector in the nominal position. To find the derivative at the bearing contact we must rotate the sector back to the bearing:

$$\dot{\xi}_0(\psi) = B(\psi_0-\psi)\dot{\xi}(\psi)$$

The derivative of the bearing rotation at the contact point is simply the projection of $\dot{\xi}_0(\psi)$ on the plane perpendicular to the bearing axis. This plane is spanned by the matrix

$$X_0 = \begin{pmatrix} e & 0 \\ -w & 0 \\ 0 & 1 \end{pmatrix}$$

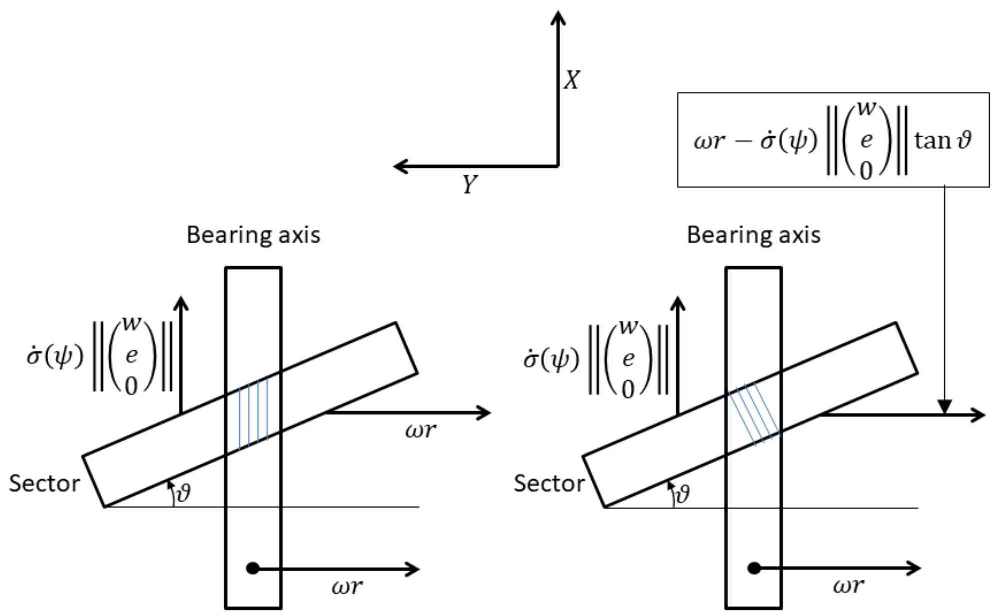
Note that the planes spanned by X_0 and X are different though they may be close. The projection on the plane perpendicular to the bearing axis is then

$$P = (X'_0 X_0)^{-1} X'_0$$

The sine of the angle ϑ between the sector tangent and the roller bearing derivative at the contact interface is then found as

$$\sin\vartheta = \frac{(I-P)\dot{\xi_0}'(\psi)}{\|\dot{\xi_0}(\psi)\|}\frac{\begin{pmatrix} w \\ e \\ 0 \end{pmatrix}}{\left\|\begin{pmatrix} w \\ e \\ 0 \end{pmatrix}\right\|}$$

From the picture below, if the motor axis surface is driven at constant nominal speed, there is a difference in tangential speed of the sector in the two cases where the grooves are parallel to the bearing axis vs. perpendicular to the sector. The formulas are in the picture without further discussion because these differences are largely hypothetical and irrelevant for a well-balanced system. If the center of mass deviates however, then the slippage could be at a nonzero angle with the bearing axis, in which case the effect would be somewhere in between the two extremes, and there could be a difference.



Design Parameters and Optimization

To find a user-friendly set of design parameters, let us summarize the variables that we used:

- λ: The observer's latitude
- d: Distance between the Dobsonian base and the center of mass (Dobsonian + base plate)
- e: Half the distance between the North bearings
- f: Vertical distance between the base plate and the bearings, also the pivot post size
- u: Horizontal distance between the North bearings and the center of mass
- w: Horizontal distance from the cone apex to the bearings
- r: Radius of the Gee disc
- q: Vertical distance from the base plate to the center of the Gee disc

- p: Horizontal distance from the cone apex to the center of the Gee disc
- m_t: Margin between the (top of the) sector and the Dobsonian base
- m_b: Margin between the (bottom of the) sector and the platform base
- s_t: Top of the sector
- s_b: Bottom of the sector
- T: Total travel time from start to reset
- ψ_T: Angle corresponding with half the run time T

From a user design perspective, any parameters directly related to the cone or Gee disc are virtually useless, because they are virtual, and cannot be seen or measured. So, let us see which variables can be eliminated and/or derived from user-friendly design parameters.

Let us first show how to derive r if w, f, λ and e are known. First, let us recall

$$q = (w - f \tan \lambda) \cos \lambda \sin \lambda$$

$$\psi_0 = \sin^{-1} \frac{q + f}{r \cos \lambda}$$

$$e = r \cos \psi_0 = r \cos \left(\sin^{-1} \frac{q + f}{r \cos \lambda} \right)$$

Clearly, r enters in a nonlinear way and cannot be solved analytically. However, it is clear that a numerical solution is straightforward by considering e in function of r, w, f and λ with r the only unknown.

Secondly, let us consider how to solve r and w if f, λ, e, d, u, m_t and T are known. First, w follows from

$$(w - u) \tan \lambda = d + m_t + s_t$$

Here, s_t is unknown and depends on T, among others. In words, the total travel time determines the sector size that affects s_t, given the constraints. This is not a simple relationship, considering the complexity of constructing the VNS curves as we have seen before. Let us sketch the dependencies involved. First, we have

$$p = (w - f \tan \lambda)(\cos \lambda)^2$$

$$q = (w - f \tan \lambda) \cos \lambda \sin \lambda$$

$$\psi_0 = \sin^{-1} \frac{q + f}{r \cos \lambda}$$

Let us define the rotation angle corresponding with half the travel time $T/2$ as ψ_T. The tip of the Gee disc section corresponding with that angle is $s(\psi_0 - \psi_T)$, where s is the Gee disc definition function defined earlier. Depending on whether f is zero or nonzero we execute a projection or milling algorithm to calculate the corresponding element of the VNS curve. The Z component is then s_t. We can then use the formula

$$\tan\lambda = \frac{d + m_t + s_t}{w - u}$$

to implement an optimization error in function of s_t. The other error function is the error in e that we used to calculate the radius r. Thus, we have a 2-dimensional error vector in function of r and w for optimization.

We have quietly ignored one detail, namely, that r is constrained by

$$r\cos\lambda \geq q + f$$

To use an unconstrained optimization algorithm it makes sense to define an auxiliary variable x implicitly as

$$r = \frac{q + f}{\cos\lambda} + x^2$$

And use x as the optimization variable instead of r. This will work fine with suitable initialization.

This completes the design of the North sector. If the design has a South sector, we can add a design parameter v as the distance between the North and South bearings. We can define $w' = w - v$ and use it instead of w, bypassing the need for the first equation that derives w from λ, u and d. The rest follows the same way, using a new bearing distance e' instead of e.

Calibrating the Completed Product

When using simple hand tools to build the VNS platform, the accuracy will be limited. Sometimes errors can happen in the translation of the idealized dimensions to components with nonzero thickness or diameter as are used in some parts of the design. Errors are not always catastrophic; so long as there is a well-defined cone axis around some cone, perhaps not the intended one, things can be salvaged by leveling (bubble level) and aligning (compass) the EQ platform appropriately.

For this, we need to do the following:

- Find the cone axis for a given position of the platform
- Find the projection of the axis on the horizontal plane and mark it on the bottom board to line up a compass with to find North with a compass
- Find the vertical leveling correction needed to point the axis to the NCP
- Find the radius of the Gee circle to determine the proper motor speed.

First, we have to take the necessary measurements:

- Slide a table against the wall. Make sure that the table top is horizontal and that the wall is vertical, using a level.
- Position the platform with the top bar of the T parallel to the wall.

- Put a photographic tripod on the platform with a laser collimator or laser pen attached to its head.
- Rotate the platform back and forth to discover the spots where the laser dot projection on the wall and table does not move anymore. Mark these points on the wall and the table with pencil on removable Scotch tape. The cone axis goes through these points.
- Using a level, find the intersection with the table by going down vertically from the stationary point on the wall. Connect this point with the stationary point on the table, and mark the connecting line on the bottom board of the platform.
- Attach a compass parallel to this line to the top of the bottom board. Use a compass that has a scale for the magnetic deviation. Find this deviation at your location on the internet, and position the platform accordingly every time you use the platform.
- Mark the position of the bubble on the bubble level, should it not be centered. When using the platform, make sure the bubble is in the same position.
- Measure the distance γ between the table stationary point and each of the front VNS bearing contacts.
- Measure the height of the wall stationary point to the table ω and that of the table stationary point to the wall τ.
- Define the vector Y_1 as the one that points from the table stationary point to the wall stationary point.
- Measure the height h above the table of the front VNS bearing contacts to the table as well.
- Measure the distance $2e$ between the bearing contacts, just like how we defined e earlier.
- Define $\psi = \sin^{-1}\frac{h}{\gamma}$ as the angle of the connecting line between each of the two stationary points and the table. This should be close to the latitude of the observer.
- Define $\phi = \sin^{-1}\frac{e}{\gamma\cos\psi}$ as half the angle between the connecting lines from the table stationary point to the bearing contacts as seen from above, just like how we defined ϕ earlier.

Then, with the coordinate system defined the same way as earlier, we define

$$Y_1 = \begin{pmatrix} -\tau \\ 0 \\ \omega \end{pmatrix}$$

$$Y_2 = \begin{pmatrix} -\gamma\cos\psi\cos\phi \\ e \\ h \end{pmatrix}$$

We define a singular value decomposition to find the plane perpendicular to Y_1, using some Scilab-like matrix notation in the following,

$$[U, S, V] = SVD(Y_1)$$

Then the last 2 columns of U span that plane. Thus, the intersection of Y_1 with that plane can be found by solving

$$Y_2 = \begin{pmatrix} Y_1 & -U(:,2:3) \end{pmatrix} \vartheta$$

This can be done using least squares or inversion. The solution is then

$$p = Y_2 + U(:,2:3)\vartheta(1:2) = Y_1\vartheta(3)$$

And the radius $r = \|Y_2 - p\|$. This completes the verification of the design, and the information that we need to make the corrections.